Gravitational Shapiro Delay as Constitutive Retardation

in an Effective Hyper-Elastic Vacuum Continuum

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Abstract

We derive the gravitational (Shapiro) radar time delay as a constitutive optical-retardation effect in an effective hyper-elastic continuum model of the vacuum. Using a weak-field refractive-index profile previously shown to reproduce the full Schwarzschild light-deflection angle via density scaling and anisotropic shear-modulus softening [10], we compute the excess light-travel time for a signal passing near a compact mass. The resulting delay contains the standard logarithmic term and matches the familiar weak-field Schwarzschild prediction at first order in $GM/(rc^2)$. This result closes the lensing-timing consistency loop in the static weak-field regime for the constitutive-optics approach: a single effective index profile accounts for both bending (lensing) and lag (time delay) within the regime of geometric optics and static spherical symmetry.

1 Introduction

The Shapiro time delay [1] — the excess travel time of a radar signal passing near a gravitating body — is one of the four classical tests of general relativity and has been measured with sub-percent precision in the solar system [2]. In the standard treatment, the delay arises from the curvature of spacetime slowing the coordinate speed of light in a gravitational potential.

An alternative phenomenological picture treats the vacuum as an effective elastic continuum with position-dependent material properties. In this constitutive framework, a gravitating mass induces local changes in vacuum density and shear modulus, which reduce the local phase speed of transverse waves and thereby introduce an effective refractive index $n(r) > 1$. This approach has been shown to reproduce the full Schwarzschild gravitational lensing deflection angle [10] and the dynamic enforcement of Lorentz invariance for translating wave packets [11].

A natural question is whether the same constitutive index profile that reproduces light bending is consistent with timing observables. An index profile could in principle be tuned to deflect rays correctly while yielding an incorrect travel-time functional. The present paper closes this loophole in the weak-field static regime: we show that the index $n(r) \approx 1 + 2GM/(rc^2)$ derived from the lensing analysis also produces the correct leading-order Shapiro time delay directly from the optical-path integral. This is a demonstration of internal consistency — not an independent derivation of the delay from first principles — and is presented as such throughout.

2 Scope and Historical Context

2.1 Regime of Validity

This paper applies to weak fields ($GM/rc^2 \ll 1$), static spherically symmetric sources, the geometric-optics (ray) approximation, and first-order evaluation with a straight-line path approximation. It does not modify the kinematics of Special Relativity, does not claim that spacetime curvature is ruled out, and makes no statement about strong-field or dynamical-spacetime regimes (black holes, gravitational waves, etc.). The result is purely phenomenological: given a constitutive index profile $n(r)$, we compute the associated travel-time delay in geometric optics and compare with the known Shapiro formula.

2.2 Historical Precedents for the Effective-Medium Equivalence

The equivalence between weak-field gravitational optics and an effective medium with $n(r) > 1$ has a long history. Eddington's 1919 analysis of light deflection by the Sun [4] implicitly employed an effective speed reduction $c(r) \approx c_0(1 - GM/rc^2)$, equivalent to a refractive index $n(r) \approx 1 + GM/rc^2$ in the linearized approximation; the full GR factor of two arises from the inclusion of spatial curvature. The formal mathematical equivalence between curved spacetime and a flat spacetime with position-dependent dielectric properties was established by Gordon (1923) [5], whose optical metric showed that Maxwell's equations in curved spacetime can be recast as propagation in an effective anisotropic medium. This framework was extended by Plebanski (1960) [6], who derived the dielectric tensor analogy for gravitational fields in full generality, and by de Felice (1971) [7], who developed the "gravitational refractive index" formalism and applied it systematically to light propagation near compact objects.

The present contribution is not the effective-medium equivalence itself, which is well-established, but the demonstration that a specific constitutive scaling—density increase and anisotropic shear-modulus softening derived from an elastic-continuum picture—is internally consistent across two independent classical observables (angular deflection and travel-time delay) without additional free parameters or tuning.

3 Constitutive Refractive Index in the Weak Field

In an effective elastic-continuum description, the vacuum is characterized by an asymptotic density ρ_0 and shear modulus μ_0, which define an asymptotic transverse wave speed $c_0 \equiv \sqrt{\mu_0/\rho_0}$, identified with the measured vacuum speed of light. In the presence of a gravitational source, local perturbations in density and shear modulus alter the wave speed. The refractive index relative to the asymptotic value is

$$n(\mathbf{x}) \equiv \frac{c_0}{v(\mathbf{x})} = \sqrt{\frac{\rho(\mathbf{x})}{\rho_0}}\sqrt{\frac{\mu_0}{\mu(\mathbf{x})}}, \tag{1}$$

where $v(\mathbf{x}) = \sqrt{\mu(\mathbf{x})/\rho(\mathbf{x})}$ is the local shear-wave speed.

For a static, spherically symmetric weak field sourced by a compact mass M, a first-order constitutive scaling (density increase plus transverse shear-modulus softening) yields an effective index [10]

$$n(r) \approx 1 + \frac{2GM}{rc^2}, \qquad \frac{GM}{rc^2} \ll 1. \tag{2}$$

The factor of two arises from the combination of radial density enhancement and anisotropic shear-modulus softening in the radial direction; "anisotropic softening" refers to the directional dependence of the effective elastic response due to radial strain in the substrate, as detailed in [10]. This is the same functional form as the coordinate speed of light in the Schwarzschild metric in isotropic coordinates, $c(r)/c_0 \approx 1 - 2GM/(rc^2)$, to first order in $GM/(rc^2)$. We treat Eq. (2) as a constitutive input imported from the lensing analysis and test it against an independent classical observable: the radar time delay.

4 Optical-Path Time and Excess Delay

4.1 Fermat Principle and Time-Delay Functional

In geometric optics, the travel time of a signal along a path Γ follows from Fermat's principle:

$$t = \frac{1}{c_0} \int_\Gamma n(\mathbf{x})\, ds, \tag{3}$$

where ds is the Euclidean line element along the ray. The excess time delay relative to propagation in flat vacuum ($n \to 1$) is

$$\Delta t \equiv t - t_0 = \frac{1}{c_0} \int_\Gamma \big[n(\mathbf{x}) - 1\big]\, ds. \tag{4}$$

This formulation connects directly to the standard gravitational lensing time-delay formalism [8, 9], where the total delay is decomposed into a geometric term (path length deviation) and a potential term (Shapiro delay) via the Fermat potential. The integral in Eq. (4) corresponds to the potential (Shapiro) component; the geometric component arises at second order in the straight-line approximation used below.

4.2 Straight-Line Approximation and Geometry

In the weak-field limit we approximate the ray trajectory by an unperturbed straight line with impact parameter b (closest approach distance). This approximation is justified because the gravitational deflection angle $\alpha \sim 4GM/(bc^2)$ introduces path deviations of order $\alpha \cdot b \ln(r/b) \sim (4GM/c^2)\ln(r/b)$, which enter the time-delay integral at second order in $GM/(rc^2)$. For solar-system parameters ($M = M_\odot$, $b \sim R_\odot$), the deflection angle is $\alpha \sim 10^{-6}$ rad, confirming the approximation is negligible at the precision level of this calculation.

Letting x parametrize the coordinate along the asymptotic propagation direction with the mass at the origin,

$$r(x) = \sqrt{x^2 + b^2}, \qquad ds \approx dx. \tag{5}$$

Substituting Eqs. (2) and (5) into Eq. (4) gives, to first order,

$$\Delta t \approx \frac{1}{c_0} \int_{x_E}^{x_R} \frac{2GM}{c^2\, r(x)}\, dx = \frac{2GM}{c_0 c^2} \int_{x_E}^{x_R} \frac{dx}{\sqrt{x^2 + b^2}}, \tag{6}$$

where x_E and x_R are the emitter and receiver positions along the ray line. Since c_0 is the asymptotic vacuum wave speed identified with the measured speed of light, we have $c_0 = c$ and the prefactor simplifies to $2GM/c^3$.

4.3 Evaluation and Logarithmic Form

The integral in Eq. (6) evaluates analytically:

$$\int \frac{dx}{\sqrt{x^2+b^2}} = \ln\Big(x+\sqrt{x^2+b^2}\Big) + C, \tag{7}$$

yielding

$$\Delta t \approx \frac{2GM}{c^3}\left[\ln\left(x+\sqrt{x^2+b^2}\right)\right]_{x_E}^{x_R}. \tag{8}$$

4.4 Asymptotic (Radar-Ranging) Form

When the emitter and receiver are far from the mass relative to b, we set $r_E \equiv \sqrt{x_E^2+b^2} \approx |x_E|$ and $r_R \equiv \sqrt{x_R^2+b^2} \approx |x_R|$. For $x_E < 0$ (emitter on the far side of the mass) and $x_R > 0$,

$$x_R + \sqrt{x_R^2+b^2} \approx 2r_R, \qquad x_E + \sqrt{x_E^2+b^2} \approx \frac{b^2}{2r_E}, \tag{9}$$

giving the familiar logarithmic expression for a one-way signal

$$\boxed{\Delta t \approx \frac{2GM}{c^3}\,\ln\left(\frac{4r_E r_R}{b^2}\right)} \tag{10}$$

to first order in $GM/(rc^2)$ under the straight-line approximation, expressed in isotropic coordinates consistent with the constitutive index form (2).

Equation (10) gives the one-way delay for a signal traveling from emitter to receiver. A standard round-trip radar measurement accumulates twice this delay; the factor of two there arises from the two-way path, not from the constitutive model. This result matches the standard weak-field Shapiro time delay [1, 3] (up to conventional geometric parametrizations of b and the endpoint radii).

5 Discussion

5.1 Closing the Lensing-Timing Consistency Loop

A standard critique of medium-based or refractive reformulations of gravity is that reproducing light bending alone is insufficient: an effective index profile could be tuned to deflect rays at the correct angle while yielding an incorrect optical-path travel time. The present result addresses this concern directly. Given only the constitutive index $n(r) \approx 1 + 2GM/(rc^2)$ — the same profile that reproduces the Schwarzschild deflection angle [10] — the optical-path integral (3) yields the correct leading-order Shapiro delay (10) without additional free parameters or tuning.

This consistency is not automatic. Bending depends on the transverse gradient of $n(r)$, whereas the time delay depends on the line integral of $n(r)-1$ along the propagation path. The fact that both observables follow from a single weak-field constitutive scaling strengthens the internal coherence of the elastic-continuum picture in the static, weak-field regime.

5.2 Interpretive Note and Scope of the Result

Within the stated regime (weak field, static symmetry, geometric optics, first order), this result is numerically identical to the GR prediction and constitutes no independent experimental test of the substrate model. Its value is demonstrative: the constitutive approach is internally coherent across two classical observables that depend on different functionals of $n(r)$ — its transverse gradient (bending) and its line integral (time delay) — using a single profile with no additional assumptions.

This paper does not claim uniqueness of the constitutive description. The same weak-field phenomenology is equally well described by the standard metric formalism (coordinate light-speed reduction in the Schwarzschild potential). The effective-medium equivalence is well-established in the literature [5, 6, 7]; the point of the present series is to show that a specific physically motivated constitutive scaling — derived from elastic-continuum assumptions rather than fitted to GR outputs — is consistent across multiple independent classical observables.

5.3 Discriminating Predictions and Higher-Order Terms

The present calculation is restricted to first order in $GM/(rc^2)$. At this order, the constitutive result is numerically identical to GR and makes no prediction distinguishable by experiment within the stated regime. Discriminating predictions require second-order calculations.

The standard GR second-order Shapiro delay includes post-Newtonian corrections $\sim (GM/rc^2)^2$ with coefficients fixed by the Schwarzschild geometry. The substrate model's second-order behavior depends on the constitutive laws for $\rho(r)$ and $\mu(r)$ at $O\big((GM/rc^2)^2\big)$: if the elastic-deformation scaling laws produce specific second-order index corrections, the resulting delay coefficients may differ from the GR values. Calculating these terms explicitly—from the substrate's elastic energy functional rather than by matching to GR—is the primary path by which the constitutive approach could yield experimentally testable deviations. This is identified as the most important item in the future-work programme.

6 Conclusions

We have shown that the effective refractive index $n(r) \approx 1+2GM/(rc^2)$, derived from a constitutive hyper-elastic continuum model of the vacuum, produces the correct leading-order Shapiro radar time delay via the standard optical-path integral. The key results are:

1. The travel-time functional in the constitutive framework is $t = (1/c_0)\int n\,ds$, with no additional assumptions beyond geometric optics.
2. For $n(r) \approx 1+2GM/(rc^2)$, the excess one-way delay takes the logarithmic form $\Delta t \approx (2GM/c^3)\ln\big(4r_E r_R/b^2\big)$, consistent with the standard Shapiro formula [1].
3. The same single index profile accounts for both gravitational lensing (angular deflection) and the Shapiro time delay in the static weak-field regime, closing the lensing-timing consistency loop for the constitutive-optics approach.

Future work. Natural extensions include: second-order path corrections beyond the straight-line approximation, which are the primary route to predictions distinguishable from GR; extension to non-spherical mass distributions and external shear relevant to strong gravitational lens environments; a unified Fermat-potential presentation [8] connecting the constitutive lensing and time-delay formalisms; and a joint analysis with the Lorentz-invariance stability result [11] to assess the full classical-test consistency of the constitutive picture.

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